OBLIQUE IMPACT OF A SOLID BODY ON SOILS

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The interaction of a solid body with soils has applications in various branches of engineering and has been studied by a number of authors.

A. Ya. Sagomonyan [1,2] studied the penetration of a sharp axisymmetric body into soils. B. I. Didukh [3] made use of Kh. A. Rakhmatulin's theory of unloading waves to solve shock compaction of loess soils with a plane compacter. An approximate solution of the same problem in elementary functions has been given by S. S. Grigoryan [4].

This paper examines the oblique impact on soils of a solid body which makes a parabolic (plane problem) or paraboloidal (axisymmetric body) impression in the surface of soils with the capability of significantly changing their density under compaction.

Sections 1–3 deal with the impact of a solid body on soils that constitute an elastoplastic medium with a piecewise-linear law of uniaxial deformation with an interval where σ = const.

1. A solid body with a sufficiently sloping profile (maximum angle between plane tangent to surface of body interacting with the soil and undeformed soil surface $\vartheta \sim \varepsilon$; ε; is a small parameter), on impact with the soil, has the components of initial velocity W_0 and U_0 (Fig. 1).

It is assumed that the body was not rotating prior to impact and that during its penetration into the soil its angular acceleration is negligibly small, since the corresponding moment of inertia is sufficiently large.

We make the approximate assumption that the unsteady motion of the soil takes place mainly in one direction, or, more precisely that the components of the velocity vector are of the order.

$$u \sim v \sim \varepsilon W_0, \quad w \sim W_0 \tag{1.1}$$

(it is assumed that the coefficient of friction is $f \sim \varepsilon^2$), while the derivatives of the different functions characterizing the soil compaction process are given by the estimates

$$\frac{\partial}{\partial x} \sim \frac{\partial}{\partial y} \sim \varepsilon \frac{\partial}{\partial z}. \tag{1.2}$$

Then as S. S. Grigoryan [5] has shown, one can derive a system of equations of one-dimensional motion along the z axis from the equations describing soil movement, by neglecting quantities $O(\varepsilon^2)$.

The corresponding approximate equations of continuity and motion have the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho w)}{\partial z} = 0, \quad \rho\left(\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z}\right) = \frac{\partial \sigma_{zz}}{\partial z} \tag{1.3}$$

(the subscripts zz on the stress components will in the future be omitted: $\sigma_{zz} = \sigma$).

The soil can be in only two states; elastic OA or packed, incompressible B_1BC (Fig. 2). The transition to the second state takes place abruptly at $\sigma = -p_s$ (θ is the relative deformation). Unloading in the compacted soil takes place without alteration of volume CBB_1.

Since the soil model in question is a particular case of the general model studied in [5], we can make use of equations [1.3]. Taking into account the incompressibility of the soil in the state characterized by points of the straight line B_1, BC we obtain equations which describe the motion along the z axis in the zone of compacted soil:

$$\frac{\partial w}{\partial z} = 0, \qquad \rho_1 \frac{\partial w}{\partial t} = \frac{\partial \sigma}{\partial z}. \tag{1.4}$$

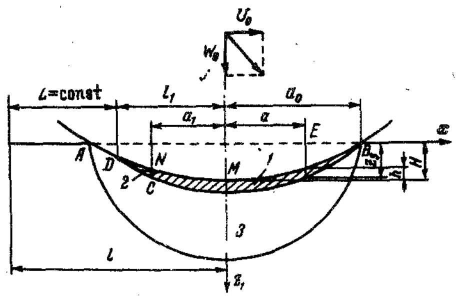

Fig. 1

It is obvious that if the component of the initial velocity W_0, normal to the soil surface, is sufficiently large, by analogy with the case of an explosion in elastoplastic soil investigated by N. V. Zvolinskii [6], the soil deformation process can be divided into five stages.

1) A compaction shock wave propagates through the undisturbed medium in the direction of the z axis (Fig. 1).

2) Under the left "stern" end of the penetrating body (in the neighborhood of point A) the shock wave velocity is first equal to and then less than c_0, the propagation velocity of the longitudinal elastic waves. The shock wave begins to emit an elastic wave; soil compaction continues; the right boundary of the region of emission of the elastic wave propagates to the right.

3) For a certain decrease in the normal velocity component, soil compaction ceases under the stern end of the body, i.e., an elastic wave will be propagated through the undisturbed medium, followed by the compacted zone; the boundary is a contact discontinuity.

4) Subsequent decrease in the normal velocity component leads to separation of the soil from the stern part of the body surface (if the initial value of the drift velocity U_0 is commensurable with the initial value of the normal component of velocity W_0).

5) Since unloading of the compacted soil takes place without alteration in volume, at the same time as the body comes to rest, in the case of direct impact, the compacted zone will stop, and from it there will separate the trailing front of the elastic wave. In the case of oblique impact a ricochet is possible.

In the elastic zone, as is customary, we ignore the convection part of the aceeleration ($\partial w/\partial t = dw/dt$) and assume that the motion, as in the zone of compacted soil, is practically one-dimensional. Then, in the plane wave between the stress $\sigma_2(\sigma_{zz})$ and the rate of elastic displacement w_2 the relation

$$\sigma_2 = -\rho_0 c_0 w_2 \tag{1.5}$$

holds, where ρ_0 is the initial soil density, c_0 is the propagation velocity of the longitudinal waves.

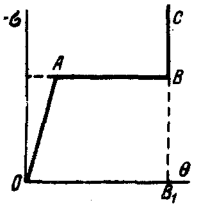

Fig. 2

Below, we will confine ourselves to a study of the more interesting second and third stages, i.e., we will assume that W_0 is moderate, and that during the whole period of its existence the entire surface of the shock wave emits an elastic wave.

Then boundary conditions can be written in the following form (see Fig. 1, where AD is the region of elastic strains, DC is the contact discontinuity front, and CB is the shock wave): at the shock wave

$$\begin{aligned} w_1 &= \frac{\rho_0}{\rho_1} w_2 + \left(1 - \frac{\rho_0}{\rho_1}\right)\frac{\partial z_s}{\partial t}, \\ -\sigma_1 &= -\sigma_2 + \frac{\rho_1^2}{\rho_0}\left(1 - \frac{\rho_0}{\rho_1}\right)\left(\frac{\partial z_s}{\partial t} - w_2\right)^2; \end{aligned} \tag{1.6}$$

at the contact discontinuity front

$$w_1 = w_2 = \frac{\partial z_s}{\partial t}, \qquad \sigma_1 = \sigma_2.$$

Here σ_1 and ω_1 correspond to compaction zones 1 and 2; σ_2 and ω_2 to the elastic zone 3; and ρ_1 is the density of the compacted soil.

The system considered can only be realized for the additional condition at the shock wave [6–8]

$$\sigma = -p. \tag{1.7}$$

The boundary conditions on the body surface for the plane problem and for an axisymmetric body have, respectively, the forms

$$w_1 = H^{\cdot} \pm l^{\cdot} \frac{dh}{da}, \qquad w_1 = H^{\cdot} + l^{\cdot} \frac{dh}{da}\cos\varphi. \tag{1.8}$$

Here the dot denotes differentiation with respect to time, H and l are coordinates of the apex M of the body relative to a fixed point on the soil surface, φ is the angle in the horizontal plane measured from the x axis; $h = h(a)$ is the body contour equation, the upper (lower) signs corresponding to points on the right (left) of the axis of symmetry (Fig. 1).

Beginning at a definite moment of time the contact surface between body and soil will be made up of three regions: a right region BN, under which compaction of the soil continues; a middle region ND, under which compaction has ceased, and a "stern" region DA under which there is no compacted soil.

Since, in view of relations (1.5) and (1.7), there is a zone with constant parameters $\sigma = -p_s$, $w_2 = w_s$, $w_s = p_s/(\rho_0 c_0)$ between the shock front and the leading front of the elastic wave, soil packing takes place under that portion of the contact surface where $w_1 > w_s$. Thus, the boundary between the first and second regions is determined by the condition

$$w = w_s. \tag{1.9}$$

The ratio $H^{\cdot}/l^{\cdot}$ decreases monotonically. In consequence, projection of the boundary between the second and third regions onto the undeformed soil surface does not change in the stationary coordinate system in the case of the plane problem, and only increases with penetration in the case of an axisymmetric body (this projection consists of points at which the soil particles attain a velocity $w = w_s$ at the moment of contact with the body).

Solving Eqs. (1.4) and (1.5) in conjunction with the boundary conditions (1.6) (1.7) and (1.8), we obtain the stress distribution $\sigma(\sigma_{zz})$ and Δ, the thickness of the zone of compacted soil along the surface of the body in regions 1, 2, and 3, respectively. Thus, for the plane problem

$$\begin{gathered} -\sigma^{(1)} = p_s + \frac{\rho_0}{\xi}\left[f(a, a_0)\left(H^{\cdot\cdot} \pm \frac{dh}{da} l^{\cdot\cdot} - \frac{d^2h}{da^2} l^{\cdot 2}\right) + \right. \\ \left. + \left(H^{\cdot} \pm \frac{dh}{da} l^{\cdot} - w_s\right)^2\right], \qquad -\sigma^{(2)} = \rho_0 c_0 \left(H^{\cdot} \pm \frac{dh}{da} l^{\cdot}\right) + \\ + \frac{\rho_0}{\xi} f(a_1^{\circ}, a_0^{\circ})\left(H^{\cdot\cdot} \pm \frac{dh}{da} l^{\cdot\cdot} - \frac{d^2h}{da^2} l^{\cdot 2}\right)\left(\xi = 1 - \frac{\rho_0}{\rho_1}\right) - \\ -\sigma^{(3)} = \rho_0 c_0 \left(H^{\cdot} \pm \frac{dh}{da} l^{\cdot}\right), \quad \Delta^{(1)}(a, a_0) = \frac{f(a, a_0)}{\xi}, \\ \Delta^{(2)}(a, a_0) = \frac{f(a_1^{\circ}, a_0^{\circ})}{\xi}, \ \Delta^{(3)}(a, a_0) = 0, \\ f(a, a_0) = h(a_0) - h(a) - w_s[t(a_0) - t(a')], \\ a' = a \pm [l(a_0) - l(a')]. \end{gathered} \tag{1.10}$$

Here $t(a)$ is the time taken for the width of the impression to increase from 0 to $2a$; $2a^1$ represents the width of the impression at the moment when the body touched the soil at point E; the superscript ° is used to denote parameters corresponding to the moment of passage of the projection on the horizontal plane of the boundary between zones 1 and 2 through a point at a distance $L = l_1 - a$ from the fixed point D.

Thus, the thickness of the compacted soil zone $\Delta < [h(a_0) - h(a)]\xi^{-1}$ and, consequently, due to the soil's ability substantially to change its density under deformation ($\xi \sim 1$), the compacted zone is a thin layer. This means that the ratio of l_z in the direction of the z axis to l_{xy} in directions perpendicular to the z axis is small, $z(l_z \sim \varepsilon l_{xy})$, which leads to estimates (1.1) and (1.2) [5]. A similar explanation can be found for the one-dimensional nature of soil motion in the elastic zone during the initial stage of penetration, when zone 3 (Fig. 1) is comparatively narrow. With further penetration one-dimensionality is assumed due to special properties of the soil. It should be noted, moreover, that at the most interesting stage, when the ratio

$w_S/H^{\cdot}$ is small, the influence of the elastic strains on the process is a second-order effect.

The equations of motion of a penetrating body with mass m and the initial conditions are written in the form

$$mH^{\cdot\cdot} = b\int_{-a_*}^{a_0} \sigma da,$$

$$ml^{\cdot\cdot} = b\int_0^{a_*} \sigma \frac{dh}{da}\, da - b\int_{-a_0}^{0} \sigma \frac{dh}{da}\, da + fb\int_{-a_0}^{a_0} \sigma da\,, \qquad (1.11)$$

$$H^{\cdot} = W_0, \quad l^{\cdot} = U_0 \quad \text{for} \quad H = 0. \qquad (1.12)$$

Here b is the width of the penetrating body.

2. Let $h(a) = a^2/2R$. Since only shallow penetration depths H are involved, it can be assumed that this contour is approximately a circle of radius R. We select the ratio w_S/W_0 as the small parameter ε and transform to dimensionless quantities

$$H^{\cdot} = W_0 W, \quad l^{\cdot} = W_0 U, \quad a = \alpha a_*, \quad b = \beta a_*,$$
$$p_* = \Pi \rho_0 W_0^2 \quad t = \varepsilon\xi\tau a_* / W_0, \quad \mu = 4\varepsilon a_*^3 \rho_0 / m,$$
$$a_* = 2\varepsilon\xi R, \quad f = \varepsilon^2 f_1.$$

Since the penetrating surface slopes, it can be taken that $\alpha \sim \tau \sim 1$. The most interesting stage of the motion is when the vertical component $H^{\cdot}$ of the penetration velocity is commensurable with its initial value W_0 ($W \sim 1$). It is assumed that the initial values of the velocity components W_0 and U_0 are of the same order and, hence, that $U \sim 1$. It is also assumed that the body width is of same order as the characteristic dimension in the direction of the x axis ($\beta \sim 1$), while the additional mass of the soil is commensurable with the body mass ($\mu \sim 1$).

Then the equations of motion (1.11) for the period prior to the appearance of the contact discontinuity zone and the initial conditions (1.12) can be written in the form

$$\left[1 + \frac{1}{2}\beta\mu\left(\frac{2}{3}\alpha_0^3 - \varepsilon\alpha_0\tau + \varepsilon\int_0^{\alpha_0} \tau d\alpha\right)\right] W\frac{dW}{d\alpha_0} +$$
$$+ \beta\mu\alpha_0^2(W^2 - 2\varepsilon W) + \beta\mu\xi\Pi\alpha_0^2 = 0,$$
$$\frac{dU}{d\alpha_0} - \varepsilon^2 f_1 \frac{dW}{d\alpha_0} + \frac{8}{3}\varepsilon^2\xi^2\beta\mu\alpha_0^4 U = 0, \qquad (2.1)$$

$$W = 1, \quad U = U_0/W_0 \quad \text{for} \quad \alpha_0 = 0. \qquad (2.2)$$

We will seek a solution for (2.1) in the form of a power series in the small parameter ε, i.e.,

$$W = W_1 + \varepsilon W_2 + \ldots, \quad U = U_1 + \varepsilon U_2 + \ldots,$$
$$\tau = \tau_2 + \varepsilon\tau_3 + \ldots. \qquad (2.3)$$

Substituting (2.3) in (2.1), we get the first-approximation equations

$$\left(1 + \frac{1}{3}\beta\mu\alpha_0^3\right) W_1 \frac{dW_1}{d\alpha_0} + \beta\mu\alpha_0^2 W_1^2 + \beta\mu\alpha_0^2\xi\Pi = 0,$$
$$\frac{dU_1}{d\alpha_0} = 0, \quad W_1 = 1, \quad U_2 = U_0/W_0 \quad \text{for} \quad \alpha_0 = 0. \qquad (2.4)$$

The second-approximation equations have the form

$$\frac{d\tau_2}{d\alpha_0} = \frac{2\alpha_0}{W_1}, \quad \frac{dW_2}{d\alpha_0} + \varphi_{12}(\alpha_0) W_2 - \varphi_{22}(\alpha_0) = 0,$$
$$\frac{dU_2}{d\alpha_0} = 0, \quad W_2 = 0, \quad U_2 = 0, \quad \tau_2 = 0 \quad \text{for} \quad \alpha_0 = 0;$$
$$\varphi_{12}(\alpha_0) = \frac{2\beta\mu\alpha_0^2}{1 + {}^1/_3\,\beta\mu\alpha_0^3} + \frac{1}{W_1}\frac{dW_1}{d\alpha_0} \qquad \varphi_{22}(\alpha_0) =$$
$$= \frac{\beta\mu}{2(1 + {}^1/_3\,\beta\mu\alpha_0^3)}\left[\alpha_0\tau_2 - \int_0^{\alpha_0} \tau_2(\alpha)\, d\alpha\right]\frac{dW_1}{d\alpha_0} + \frac{2\beta\mu\alpha_0^2}{1 + {}^1/_3\,\beta\mu\alpha_0^3}. \qquad (2.5)$$

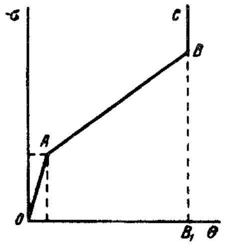

Fig. 3

The third-approximation equation is

$$\frac{dU_3}{d\alpha_0} - f_1\frac{dW_1}{d\alpha_0} + \frac{8}{3}\xi^2\beta\mu\alpha_0^4 U_1 = 0,$$
$$U_3 = 0 \quad \text{for} \quad \alpha_0 = 0. \qquad (2.6)$$

The solutions of Eqs. (2.4), (2.5) and (2.6) for the initial conditions mentioned are found in quadratures. For (2.4) we have

$$W_1 = \sqrt{(1 + \xi\Pi)(1 + {}^1/_3\,\beta\mu\alpha_0^3)^{-2} - \xi\Pi},$$
$$U_1 = U_0/W_0. \qquad (2.7)$$

For (2.5)

$$\tau_2 = \frac{2I(\delta,\omega)}{(\xi\Pi)^{1/3}(3\beta^2\mu^2)^{1/3}}, \quad W_2 = -\frac{J_1(\delta,\omega) - J_2(\delta,\omega)}{W_1(1+\omega)^2},$$
$$U_2 = 0, \quad \omega = \frac{\beta\mu\alpha_0^3}{3}, \quad \delta = \frac{1}{\xi\Pi},$$
$$I(\delta, \omega) = \int_0^{\omega} \frac{(1+\omega_1)\, d\omega_1}{\sqrt{(\delta - 2\omega_1 - \omega_1^2)\,\omega_1^{2/3}}},$$
$$J_1(\delta, \omega) = \left(\sqrt{\delta} + \frac{1}{\sqrt{\delta}}\right)[I_1(\delta, \omega) - I_2(\delta, \omega)],$$
$$J_2(\delta, \omega) = \left\{\frac{\varkappa\sqrt{\delta + 1 - \varkappa^2}}{\sqrt{\delta}} - 1 + \left(\sqrt{\delta} + \frac{1}{\sqrt{\delta}}\right) \times\right.$$
$$\left.\times\left[\arcsin\left(\frac{1}{\sqrt{1+\delta}}\varkappa\right) - \arcsin\frac{1}{\sqrt{1+\delta}}\right]\right\},$$
$$\varkappa = 1 + \omega \qquad I_1(\delta, \omega) = \int_0^{\omega} \frac{I(\delta, \omega_1)\,\omega_1^{1/3}\, d\omega_1}{(1+\omega_1)^2},$$
$$I_2(\delta, \omega) = \int_0^{\omega} \frac{I_0(\delta, \omega)\, d\omega_1}{(1+\omega_1)^2} \qquad I_0(\delta, \omega) = \int_0^{\omega} \frac{I(\delta, \omega_1)\, d\omega_1}{\omega_1^{2/3}}. \qquad (2.8)$$

For (2.6)

$$U_3 = -f_1(1 - W_1) - {}^8/_3\,\xi^2\beta\mu\alpha_0^5 U_0 / W_0. \qquad (2.9)$$

The components of the force acting on the penetrating body are

$$P_z = \frac{1}{2\varepsilon\xi}\frac{mW_0^2}{\alpha_0 a_*}\left\{\frac{\varepsilon\beta\mu}{2(1+{}^1/_3\,\beta\mu\alpha_0^3)}\left[\left(\alpha_0\tau_2 - \int\limits_0^{\alpha_0}\tau_2(\alpha)\,d\alpha\right)\frac{dW_1}{d\alpha_0} + \right.\right.$$

$$\left.\left. + 4\alpha_0^2(1 - W_2)\right] + \frac{dW_1}{d\alpha_0}\right\}W_1,$$

$$P_x = \frac{1}{2}\frac{\varepsilon}{\xi}\frac{mW_0^2}{\alpha_0 a_*}\left(f_1\frac{dW_1}{d\alpha_0} - \frac{8}{3}\xi^2\beta\mu\alpha_0^4\frac{U_0}{W_0}\right)W_1. \quad (2.10)$$

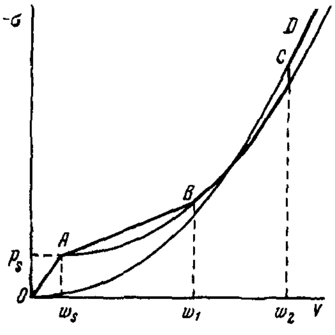

Fig. 4

Compaction of the soil under the stern end of the body ceases when the soil velocity in the neighborhood of point A equals w_s or [see(1.8)] when the dimensionless width of the impression equals

$$2\alpha_1 = \frac{W-\varepsilon}{\varepsilon\xi U}. \quad (2.11)$$

From this moment and up to the commencement of separation of the body from the soil at the stern end, its motion will be described by the system of equations

$$\left\{1 + \frac{\beta\mu}{4}\left[(\alpha_0+\alpha_1)(\alpha_0^2 - \varepsilon\tau) - \frac{\alpha_0^3+\alpha_1^3}{3} + \right.\right.$$

$$\left.\left. + \int\limits_0^{\lambda-\alpha_1} f_0(\alpha)\,d\alpha - \varepsilon\left(\int\limits_0^{\alpha_0}\tau(\alpha)\,d\alpha + \int\limits_0^{\alpha_1}\tau(\alpha)\,d\alpha\right)\right]\right\}W\frac{dW}{d\alpha_0} +$$

$$+ \frac{\beta\mu(\alpha_0+\alpha_1)\alpha_0 W^2}{2} + \frac{\beta\mu[\xi\zeta(\alpha_0-\alpha_1)\alpha_0 - 2\varepsilon(\alpha_0+\alpha_1)\alpha_0]W}{2} +$$

$$+ \beta\mu\varepsilon\xi\alpha_0(\alpha_0^2-\alpha_1^2)WU - \frac{\beta\mu\varepsilon\xi\alpha_0(\alpha_0^2-\alpha_1^2)(2+\xi\zeta)U}{2} +$$

$$+ \frac{\beta\mu\xi\Pi\alpha_0(\alpha_0+\alpha_1)}{2} = 0, \quad W\frac{dU}{d\alpha_0} + \left\{\frac{\beta\mu}{4}\left[\frac{\varepsilon\xi(\alpha_0^2-\alpha_1^2)^2}{2} - \right.\right.$$

$$- \varepsilon^2\xi(\alpha_0^2-\alpha_1^2)\tau - 2\varepsilon\xi\lambda\int\limits_0^{\lambda-\alpha_1} f_0(\alpha)\,d\alpha + 2\varepsilon\xi\int\limits_0^{\lambda-\alpha_0}\alpha f_0(\alpha)\,d\alpha +$$

$$\left.\left. + 2\varepsilon^2\xi\left(\int\limits_0^{\alpha_0}\alpha\tau(\alpha)\,d\alpha - \int\limits_0^{\alpha_1}\alpha\tau(\alpha)\,d\alpha\right)\right] - \varepsilon^2 f_1\right\}W\frac{dW}{d\alpha_0} +$$

$$+ \frac{\beta\mu\varepsilon\xi\alpha_0(\alpha_0^2-\alpha_1^2)W^2}{2} - \frac{\beta\mu(2\varepsilon^2\xi+\varepsilon\xi^2\zeta)\alpha_0(\alpha_0^2-\alpha_1^2)W}{2} +$$

$$+ \frac{4\beta\mu\varepsilon^2\xi^2\alpha_0(\alpha_0^3+\alpha_1^3)WU}{3} + \frac{2\beta\mu\varepsilon^2\xi^3\zeta\alpha_0(\alpha_0^3-\alpha_1^3)U}{3} +$$

$$+ \frac{\beta\mu\varepsilon\xi^2\alpha_0(\alpha_0^2-\alpha_1^2)\Pi}{2} = 0, \quad \xi = \frac{c_0}{W_0} \quad \frac{d\tau}{d\alpha_0} = \frac{2\alpha_0}{W},$$

$$\frac{d\lambda}{d\alpha_0} = \frac{2\alpha_0 U}{W}, \quad \lambda = \frac{l_1}{a_*}, \quad f_0(\lambda-\alpha) = f(\alpha_1^\circ, \alpha_0^\circ). \quad (2.12)$$

The nonlinear system of Eqs. (2.12) is solved by numerical methods. The initial values of W, U and τ are found from the calculation of the preceding stage. The quantity λ equals zero at the commencement of stage considered. The dimensionless coordinate α_1 of the zone boundary is found from (2.11).

3. Oblique impact of an axisymmetric body can be examined in the same way. The first stage of motion (up to cessation of soil compaction at the "stern" end) for $h = a^2/2R$ is described by the equations

$$\left[1 + \mu_1\left[\frac{1}{2}\chi^2 - \varepsilon\chi\tau + \varepsilon\int\limits_0^{\chi}\tau(\chi_1)\,d\chi_1\right]W\frac{dW}{d\chi} + \right.$$

$$+ \mu_1\chi(W^2 - 2\varepsilon W) + \mu_1\xi\Pi\chi = 0,$$

$$\frac{dU}{d\chi} - \varepsilon^2 f_1\frac{dW}{d\chi} + 2\mu_1\varepsilon^2\xi^2\chi U = 0, \qquad \frac{d\tau}{d\chi} = \frac{1}{W},$$

$$W = 1, \quad U = \frac{U_0}{W_0}, \quad \tau = 0 \quad \text{at} \quad \chi = 0.$$

$$\left(\chi = \frac{H}{\varepsilon\xi a_*}, \quad \mu_1 = \frac{\pi\varepsilon\rho_0 a_*^3}{m}\right). \quad (3.1)$$

Substituting (2.3) in (3.1) and assuming that for the dimensionless quantities we have estimates similar to those in 2, we obtain:
first-approximation equations

$$\left(1 + \frac{\mu_1\chi^2}{2}\right)W_1\frac{dW_1}{d\chi} + \mu_1\chi W_1^2 + \mu_1\xi\Pi\chi = 0,$$

$$\frac{dU_1}{d\chi} = 0 \quad W_1 = 1, \quad U_1 = U_0/W_0 \quad \text{at} \quad \chi = 0; \quad (3.2)$$

second-approximation equations

$$\frac{d\tau_2}{d\chi} = \frac{1}{W_1}, \qquad \frac{dW_2}{d\chi} + \Psi_{12}(\chi)W_2 + \Psi_{22}(\chi) = 0,$$

$$\frac{dU_2}{d\chi} = 0, \quad \tau_2 = 0, \quad W_2 = 0, \quad U_2 = 0 \quad \text{at} \quad \chi = 0,$$

$$\Psi_{12}(\chi) = \frac{2\mu_1\chi}{1+{}^1/_2\,\mu_1\chi^2} + \frac{1}{W_1}\frac{dW_1}{d\chi}, \quad \Psi_{22}(\chi) =$$

$$= -\frac{\mu_1}{1+{}^1/_2\,\mu_1\chi^2}\left\{\left[\chi\tau - \int\limits_0^{\chi}\tau(\chi_1)\,d\chi_1\right]\frac{dW_1}{d\chi} - 2\chi\right\}; \quad (3.3)$$

third-approximation equations

$$\frac{dU_3}{d\chi} - f_1\frac{dW_1}{d\chi} + 2\mu_1\xi^2\chi U_1 = 0, \quad U_3 = 0 \quad \text{at} \quad \chi = 0. \quad (3.4)$$

The equations of the successive approximations are solved in quadratures,

$$W_1 = \left[\frac{1+\xi\Pi}{(1+{}^1/_2\,\mu_1\chi^2)^2} - \xi\Pi\right]^{1/2}, \qquad U_1 = \frac{U_0}{W_0},$$

$$\tau_2 = \frac{I_3(\delta,\omega)}{\sqrt{2\mu_1}}, \qquad U_2 = 0,$$

$$I_3(\delta,\omega) = \sqrt{\delta}\int\limits_0^{\omega}\frac{(1+\omega_1)\,d\omega_1}{\sqrt{(\delta - 2\omega_1 - \omega_1^2)\,\omega_1}},$$

$$W_2 = -\Phi_{22}(\chi)\exp[-\Phi_{12}(\chi)], \quad U_3 = -f_1(1-W_1) -$$

$$- \frac{\mu_1\xi^2\chi^2 U_0}{W_0}, \quad \left(\Phi_{12}(\chi) = \int\limits_0^{\chi}\Psi_{12}(\chi_1)\,d\chi_1, \quad \Phi_{22}(\chi) = \right.$$

$$\left. = \int\limits_0^{\chi}\Psi_{22}(\chi_1)\exp[\Phi_{12}(\chi_1)\,d\chi_1]\,d\chi_1\right). \quad (3.5)$$

The components of the force acting on the penetrating body are

$$P_z = \frac{mW_0^2}{\varepsilon\xi a_*}\left\{\frac{\varepsilon\mu_1}{1+{}^1/_3\mu_1\chi^2}\left[\left(\chi\tau_2 - \int_0^{\chi}\tau_2(\chi_1)\,d\chi_1\right)\frac{dW_1}{d\chi} + 2\chi(1-W_2)\right] + \frac{dW_1}{d\chi}\right\}W_1 \quad P_x =$$

$$= \frac{\varepsilon}{\xi}\frac{mW_0^2}{a_*}\left(f_1W_1\frac{dW_1}{d\chi} - 2\mu_1\xi^2W_1\frac{U_0}{W_0}\right). \qquad (3.6)$$

The compaction of the soil under the stern part of the body ceases when the dimensionless depth of the impression χ becomes equal to the quantity

$$\chi_1 = \left(\frac{W-\varepsilon}{2\varepsilon\xi U}\right)^2. \qquad (3.7)$$

The subsequent imbedding of the body in the soil may be treated in the same way as for the plane problem. Direct impact ($U_0 = 0$) may be similarly considered. In this case consideration of the third stage of the process is considerably simplified, since compaction of the soil ceases simultaneously under the entire contact surface.

It should be noted that under certain conditions the picture of the process may be modified; a cavity may form near the apex of the body. The condition for initiation of separation is that stresses σ be zero at the corresponding points of the contact surface. Calculations show that for direct impact in the first approximation soil separation will take place on that portion of the surface which has penetrated the soil to a depth $H > H_1 = [\xi m/(\pi\rho_0 R)]^{1/2}$.

The soil model described will obviously be the limiting case of an elastoplastic medium with a piecewise-linear law of uniaxial deformation (Fig. 3) (it is assumed that unloading from the states represented by points on the section AB takes place without change of volume). S. S. Grigoryan [4] used another particular case of this model, where the relative soil deformation does not reach the limiting value corresponding to point B.

4. For a sufficiently high initial body velocity, the soil deformation process will, in the case, proceed as follows.

1) The shock wave compacting the soil to the limiting incompressibile state corresponding the points on BC propagates through the undisturbed medium in the direction of the z axis (Fig. 1).

2) Under the "stern" end of the body the shock wave velocity first becomes equal to, and then less than the velocity $c_0 = E/\rho_0{}^{1/2}$. (E is Yong's modulus; section OA in Fig. 3). The shock wave begins to emit an elastic wave; compaction of the soil to the limiting state continues, and the right boundary of the region of elastic wave emission moves to the right.

3) With further decrease in the velocity component normal to the soil surface the shock wave velocity under the stern end becomes equal to $c_1 = E_1/\rho_0{}^{1/2}$ (E is the modulus of plasticity); compaction of the soil to the limiting incompressible state ceases; and behind the elastic wave, moving at velocity c_0 through the region with constant parameters ($\sigma = -p_s$, $w = w_s$), the plastic wave will propagate at velocity c_1.

It is obvious that if AB slopes sufficiently and $c_1 < w_s = p_s/(\rho_0 c_0)$ the final stage will not take place, and the soil deformation process will proceed according to the scheme of § 1.

We shall examine the third stage of motion, i.e., we shall assume that the initial velocity component W_0 is sufficiently small and that limiting compaction of the soil to the incompressible state does not take place. As before, we regard the soil motion as one-dimensional, due, for example, to its high porosity [3, 4] (powdery soil). Then it can be shown that distribution of stress $\sigma(\sigma_{zz})$ over the surface of an axisymmetric body will be

$$-\sigma(a,\varphi,a_0) = \rho_0c_1[t(a_0) - t(a')]\left[H^{\cdot\cdot} + l^{\cdot\cdot}\frac{dh}{da}\cos\varphi - l^{\cdot 2}\left(\frac{d^2h}{da^2}\cos^2\varphi - \frac{1}{a}\frac{dh}{da}\sin^2\varphi\right)\right] +$$

$$+ p_s + \rho_0c_1\left(H^{\cdot} + l^{\cdot}\frac{dh}{da}\cos\varphi - \theta_s c_0\right), \quad t(a') = t(a) +$$

$$+ \frac{dt(a)}{da}[l(a_0) - l(a)]\cos\varphi + O(\varepsilon^2 a_*/W_0),$$

$$a' = \sqrt{a^2 + (\Delta l)^2 + 2\Delta la\cos\varphi} \approx a + \Delta l\cos\varphi,$$

$$\Delta l = l(a_0) - l(a') = l(a_0) - l(a) + O(\varepsilon^2 a_*). \qquad (4.1)$$

Here $-\sigma = p_s = E\theta_s$ is the elastic limit (Fig. 3, point A).

Let $h = a^2/2R$. As before, for oblique impact, we limit ourselves to the case of shallow penetration H.

We introduce the as yet undetermined small parameter ε and transform to dimensionless quantities

$$a = \alpha a_*, \quad a_* = 2\varepsilon R, \quad H^{\cdot} = W_0W, \quad l^{\cdot} = W_0U,$$

$$\mu_2 = \frac{\pi a_*{}^3\rho_0}{m\varepsilon} \quad p_s = \frac{\rho_0\Pi W_0^2}{\varepsilon}, \quad t = \frac{\varepsilon a_*\tau}{W_0},$$

$$f = f_1\varepsilon^2, \quad c_0 = \frac{\zeta_0W_0}{\varepsilon^3}, \quad c_1 = \frac{\zeta_1W_0}{\varepsilon^2}.$$

For powdery soils, such as loess [3],

$$E = 2\cdot 10^7 \ \text{kgf/m}^2, \quad c_1^2/c_0^2 = 0.05,$$

$$\theta_s = 0.003, \quad \rho_0 = 1.5\cdot 10^2 \ \text{kgf}\cdot\text{sec}^2/\text{m}^4$$

and hence, for example for $\varepsilon \sim 0.3$, $W_0 \sim 10$ m/sec, it may be taken that $\zeta_0 \sim \zeta_1 \sim \Pi \sim 1$. Repeating the arguments of Section 2, we also obtain the estimates $a \sim W \sim U \sim \tau \sim \mu_2 \sim 1$. Then the equations of motion may be written in the form

$$\left[1 + \mu_2\zeta_1\int_0^{\alpha_0}\frac{2\alpha^3\,d\alpha}{W(\alpha)}\right]W\frac{dW}{d\alpha_0} + 2\mu_2\zeta_1\alpha_0{}^3W + 2\varepsilon\mu_2\left(1 - \frac{c_1}{c_0}\right)\Pi\alpha_0{}^3 = 0,$$

$$\frac{dU}{d\alpha_0} - \varepsilon^2\left[2\mu_2\zeta_1\int_0^{\alpha_0}\lambda\frac{d\tau(\alpha)}{d\alpha}\alpha^2\,d\alpha + f_1\right]\frac{dW}{d\alpha_0} +$$

$$+ 2\varepsilon^2\mu_2\zeta_1\alpha_0{}^5\frac{U}{W} = 0, \quad \lambda = \frac{l(\alpha_0) - l(\alpha)}{a_*\varepsilon}, \quad \frac{d\tau}{d\alpha_0} = \frac{2\alpha_0}{W}, \quad (\lambda \sim 1)$$

$$W = 1, \quad U = \frac{U_0}{W_0}, \quad \tau = 0 \quad \text{for } \alpha_0 = 0. \qquad (4.2)$$

As in §2, we will seek a solution for Eq. (4.2) in the form of a series in powers of the small parameter ε. Substituting (2.3) in (4.2), we obtain the first-approximation equation

$$\left[1+\mu_2\zeta_1\int_0^{\alpha_0}\frac{2\alpha^3\,d\alpha}{W_1(\alpha)}\right]\frac{dW_1}{d\alpha_0}+2\mu_2\zeta_1\alpha_0^3=0,\quad \frac{dU_1}{d\alpha_0}=0,$$
$$W_1=1,\quad U_1=U_0/W_0\quad \text{for } \alpha_0-0. \tag{4.3}$$

We select the small parameter ε in such a way that $\mu_2\,\zeta_1=2$. Then

$$W_1=\exp(-\alpha_0^4),\qquad U_1=U_0\,/\,W_0 \tag{4.4}$$

(for an arbitrary choice of ε we have $W_1=\exp(-k\alpha_0^4)$, $k=\mu_2\zeta_1/2$).

The second-approximation equations are

$$\frac{dW_2}{d\alpha_0}+4\alpha_0^2\exp(-2\alpha_0^4)\times$$
$$\times\int_0^{\alpha_0}4\alpha^3\exp(2\alpha^4)\,W_2(\alpha)\,d\alpha+\gamma_1\alpha_0^3=0,$$
$$\frac{dU_2}{d\alpha_0}=0,\quad \frac{d\tau_2}{d\alpha_0}=2\alpha_0\exp(\alpha_0^4)\quad \left(\gamma_1=2\mu_2\left(1-\frac{c_1}{c_0}\right)\right),$$
$$W_2=0,\quad U_2=0,\quad \tau_2=0\quad \text{for } \alpha_0=0. \tag{4.5}$$

The solution of equations (4.5) has the form

$$W_2=-\frac{\gamma_1}{4}\left[2-(2+\alpha_0^4)\exp(-\alpha_0^4)\right],\qquad U_2=0,$$
$$\tau_2=2\int_0^{\alpha_0}\alpha\exp(\alpha^4)\,d\alpha. \tag{4.6}$$

The third-approximation equations are

$$\frac{dU_3}{d\alpha_0}-\left\{2\mu_2\zeta_1\frac{U_0}{W_0}\int_0^{\alpha_0}[\tau_2(\alpha_0)-\tau_2(\alpha)]\;\frac{d\tau_2(\alpha)}{d\alpha}\alpha^2\,d\alpha+f_1\right\}\frac{dW_1}{d\alpha_0}-$$
$$-2\mu_1\zeta_1\frac{U_0}{W_0}\frac{\alpha_0^5}{W_1(\alpha_0)}=0,\qquad U_3=0\quad \text{for } \alpha_0=0. \tag{4.7}$$

Thus

$$U_3=$$
$$=-4\int_0^{\alpha_0}\alpha_2^3\exp(-\alpha_2^4)\left[2\mu_2\zeta_1\frac{U_0}{W_0}\int_0^{\alpha_2}[\tau_2(\alpha_2)-\right.$$
$$-\tau_2(\alpha)]\,\frac{d\tau_2(\alpha)}{d\alpha}\,\alpha^2\,d\alpha+$$
$$\left.+f_1\right]d\alpha_2-2\mu_2\zeta_1\frac{U_0}{W_0}\int_0^{\alpha_0}\alpha^4\exp(\alpha^4)\,d\alpha. \tag{4.8}$$

The components of the force acting on the penetrating body are

$$P_z=-\frac{mW_0^2}{2\varepsilon a_*\alpha_0}\times$$
$$\times\left[4\alpha_0^3\exp(-2\alpha_0^4)+\varepsilon\exp\alpha_0^4\left(4\alpha_0^3W_2-\frac{dW_2}{d\alpha_0}\right)\right],$$
$$P_x=-\frac{\varepsilon mW_0^2}{2a_*\alpha_0}\times$$
$$\times\left\{4\alpha_0^3\exp(-2\alpha_0^4)\left[2\mu_2\zeta_1\frac{U_0}{W_0}\int_0^{\alpha_0}[\tau_2(\alpha_0)-\tau_2(\alpha)]\times\right.\right.$$
$$\left.\left.\times\frac{d\tau_2(\alpha)}{d\alpha}\alpha^2\,d\alpha+f_1\right]+2\mu_2\zeta_1\frac{U_0}{W_0}\alpha_0^5\right\}. \tag{4.9}$$

Similar expressions are obtained for the plane problem.

5. The additional mass of the soil can be ignored when dealing with the problem of direct impact of a heavy ($\mu_2\sim\varepsilon^2$) cylindrical body with a sloping blunt end on the soil considered in Section 4. In this case the soil reaction on the body will be uniquely determined by the rate of penetration. Figure 4 gives the dependence of σ, the contact surface stress, on V the rate of penetration.

$$-\sigma=\rho_0c_0V\qquad \text{on } OA$$
$$-\sigma=p_s+\rho_0c_1V\qquad \text{on } AB$$
$$-\sigma=p_s+\frac{\rho_0}{\xi}(V-w_s)^2\qquad \text{on } BC$$
$$-\sigma=\frac{\rho_0}{\xi}V^2\qquad \text{on } CD$$
$$w_s=\frac{p_s}{\rho_0c_0},\qquad w_1=\xi c_1+(1-\xi)\,w_s$$
$$w_2=\xi c_0,\qquad \xi=1-\frac{\rho_0}{\rho_1}.$$

Here ρ is the density of the fully compacted soil (B_1BC in Fig. 3).

It should be noted that with decrease in the modulus of plasticity E_1 the velocity w_1 approaches w_s. When $w_1\le w_s$, there will be no section AB (Fig. 4).

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